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The Rationale of the Mean-Standard Deviation Analysis: Comment

By G. O. BIERWAG*

In a recent paper in this *Review*, S. C. Tsiang severely criticizes the traditional portfolio analysis within the $(E\text{-}S)$ framework. Tsiang propounds two major contentions: the $(E\text{-}S)$ framework is appropriate 1) only when E is large relative to S, and 2) when the indifference curves in the $(E\text{-}S)$ half-plane have slopes of 45° or less. I have no quibble with the first of these conditions; it is relatively well-founded in the work of Kenneth Arrow, John Pratt, and Paul Samuelson and it is based on reliable principles of approximation. Tsiang, using the second contention, shows that the $(E\text{-}S)$ framework cannot support James Tobin's theory of liquidity preference. This would, indeed, be a serious defect of the $(E\text{–}S)$ analysis, if it were true. Fortunately, the second contention is quite false. In this comment, I show why this is so and I propose a substitute contention that meets much of the spirit of Tsiang's inquiry.

I

Consider an example in which an investor may invest in two assets—one riskless and the other risky. Let x_0 and x_1 be the number of dollars, respectively, so invested in each. At the end of the investment period, the initial investment of x_0+x_1 dollars grows to $x_0+\lambda x_1=R$ dollars where it is supposed that λ is a random variable with a subjective cumulative probability distribution, $F(\lambda)$. Assume that $F(\lambda)$ is well specified by two independent parameters, μ and σ, the mean and standard deviation of λ. Let $u(R)$ be a twice differentiable utility function over wealth with the property that $u'(R)>0$ and $u''(R)<0$, i.e., the investor is risk averse. Under the above assumptions, expected utility can be expressed as a function of only the mean and standard deviation of the portfolio. Letting $E=x_0+\mu x_1$ and $S=\sigma x_1$, one may write

$$(1) \qquad U(E, S) = E[u(R)]$$

The investor now faces the problem of choosing E and S by appropriately choosing x_0 and x_1. This choice is equivalent to choosing some distribution of R, subject to the fixed distribution of λ. As Myron Grove and the author have noted and developed, the expected utility in (1) has contours of constant expected utility which map into the (x_0, x_1) space. Naturally, each point on a contour in that space maps into a point on the corresponding contour in the $(E\text{-}S)$ half-plane. Consider the slope of an indifference curve in the $(E\text{-}S)$ half-plane. Taking the total derivative of $U(E, S)$ with respect to x_1, one has

$$(2) \qquad U_E\left[\frac{dx_0}{dx_1}+\mu\right]+U_S\sigma = 0$$

along a contour of constant expected utility where dx_0/dx_1 is the corresponding slope of the contour in the (x_0, x_1) space. Then, the slope of an indifference curve in the $(E\text{-}S)$ space is

$$(3) \qquad \frac{dE}{dS} = -\, U_S/U_E = \frac{\mu}{\sigma}+\frac{1}{\sigma}\frac{dx_0}{dx_1}$$

Now, if μ is large relative to σ, it is highly likely that dE/dS exceeds unity; but, of course, everything depends on dx_0/dx_1 which we should naturally expect to be negative because x_0 and x_1 are to some extent substitutable investments. However, one can easily illustrate with examples that dx_0/dx_1 need not be so negative as to imply that $dE/dS\leq 1$. For example, suppose that λ is *gamma* distributed and that $u(R)=-\exp(-R)$. The expected utility is then

$$(4) \qquad E[u(R)] = \frac{-1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha-1} \cdot\exp(-x_0-\lambda x_1-\lambda/\beta)\,d\lambda$$

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where α and β are positive parameters of the distribution of λ. Resolving the integral in (4),

$$(5) \quad E[u(R)] = -(1+\beta x_1)^{-\alpha} \exp(-x_0)$$

Holding expected utility fixed at some level and allowing variations in x_0 and x_1, it follows that

$$(6) \quad (1+\beta x_1)dx_0 + \alpha\beta dx_1 = 0$$

along an indifference curve in the (x_0, x_1) space. The slope of the indifference surface is then

$$(7) \quad \frac{dx_0}{dx_1} = \frac{-\alpha\beta}{1+\beta x_1}$$

Noting that $\beta = \sigma^2/\mu$ and $\alpha = \mu^2/\sigma^2$, (7) becomes

$$(8) \quad \frac{dx_0}{dx_1} = \frac{-\mu^2/\sigma^2}{x_1 + \dfrac{\mu}{\sigma^2}}$$

The slope is very small in absolute value when x_1 tends to be large. Referring to (3), it is clearly evident that the slope, dE/dS, may exceed unity.[1]

The problem with Tsiang's approach is that he doesn't maintain the constancy of the parameters μ and σ. Evidently the subjective distribution of λ depends on the investor's choice of x_0 and x_1. Why this should be so is not clear. He appears to be assuming that if λ is distributed according to a lognormal distribution, say, then so is R. Clearly, this is not so.

Tsiang's 45° rule was prompted by an effort to avoid Karl Borch's problem in which Bernouilli mixtures of distributions were relevant choices in an asset space. The point is that different choices of assets in portfolio theory are *not* also choices of Bernouilli mixtures.[2]

II

In portfolio theory, as Tsiang suggests, it seems reasonable to suppose that Prob $[\lambda < 0] = 0$. This assumption reflects the fact that many financial instruments have provisions for limited liability; there are stop-loss provisions in stock and commodity markets, etc. Suppose that λ, then, is so distributed. In the $(E\text{-}S)$ framework, this restriction on λ must imply that the indifference curves have an elasticity which is less than unity. The proof may proceed as follows. The marginal expected utilities for the assets must be positive.[3] That is,

$$(9) \quad E[u'(R)] = U_E > 0$$

and

$$E[u'(R)\lambda] = U_E\mu + (U_S/S)\sigma^2 x_1 > 0$$

Multiplying the first of these inequalities by x_0 and the second by x_1, so that in adding the equations together, one has

$$(10) \quad U_E E + U_S S > 0$$

from which it is clear that

$$(11) \quad \frac{dE}{dS}\frac{S}{E} = -(U_S/U_E)(S/E) < 1$$

[1] This choice of the *gamma* function is strictly for convenience. Almost any two-parameter distribution of λ works satisfactorily as long as expected utility exists. The utility function, $-\exp(-R)$, is a bounded function for $R \geq 0$, and the measure of relative risk aversion is simply R which is unbounded from above. Given this situation, one can use one of Arrow's theorems (1971) to show that the investor will always hold money with a sufficiently large initial wealth—and this is so regardless of how many independent parameters are in the probability distribution function.

[2] The von Neumann-Morgenstern axioms still hold for this situation. Although these axioms apply to "mixture sets," there is absolutely no reason to suppose that *every* problem of choice is one that requires the points of choice to be mixtures of probability distributions. The set of portfolio choices is a *subset* of an entire mixture set to which the von Neumann-Morgenstern axioms apply. Borch's problem and Tsiang's refinement of it are consequently irrelevant to the problem of portfolio selection as it is traditionally stated. Other problems of choice under uncertainty may well involve mixtures and Borch's problem may well apply, but this is not so for the portfolio theory that gives rise to a theory of liquidity preference.

[3] Grove and I (1966) consider the case in which there is no riskless asset in the portfolio. If one allows Prob$[\lambda < 0] > 0$ for each asset, it is possible to get cases in which all assets have negative marginal utilities. Such a portfolio is then "contaminated by risk." The above assumptions thus prevent contamination.

provided the investor is risk averse, $x_0>0$, and $x_1>0$. If one, therefore, assumes that a portfolio can never be contaminated, then (11) must hold. Of course, Tobin's theory of liquidity preference can still be maintained under this restriction.

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