The Rationale of the Mean-Standard Deviation Analysis: Comment

By Karl Borch*

Portfolio analysis based on mean and variance was introduced by Harry Markowitz (1952) more than twenty years ago. This method of analysis has become extremely popular, partly through Markowitz's own book (1959), and also through the work of James Tobin. The method has been severely criticized, and I am among the critics who have argued that mean-variance analysis must be seen as an intellectual exercise, useful only if it leads to insight into the real problems of portfolio management. The method has its staunch defenders, and one of the most articulate among these is S. C. Tsiang. In a recent article in this *Review*, he gives a comprehensive survey of the problem, and concludes that mean-variance analysis gives "a fair approximation—for most practical purposes."

I have no objection to this cautious conclusion, although the term "practical purposes" may be given unwarranted interpretations. Tsiang does, however, devote a considerable amount of space to a discussion of a simple classroom example which I published a few years ago, and it may be in order to spell out the ideas behind the example. It is convenient first to present the example again in a less general form than the original. Let us consider the following two probability distributions or "gambles":

Gamble A,	which will give:
	0 with probability 0.5
or	2 with probability 0.5
Here:	Mean = 1 and Variance = 1

Gamble B,	which will give:
	−2 with probability 0.2
or	3 with probability 0.8
Here:	Mean = 2 and Variance = 4

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I will not question the rationality of a person who considers the two gambles as equally attractive. If, however, he says that the gambles are equivalent *because* the mean-variance pairs {1, 1} and {2, 4} are equivalent, I will present him with:

Gamble C,	which will give:
	0 with probability 0.5
	4 with probability 0.5

This gamble is clearly more attractive than *A*, and hence it should also be more attractive than *B*. The gambles *B* and *C* do, however, both have the mean-variance pair {2, 4}, and this should imply that they are equivalent.

For any two gambles of this form, alleged to be equivalent, we can construct a third gamble of the same form which:

1) is clearly superior to one of the original gambles
2) has the same mean-variance pair as the other original gamble

This proves that a consistent preference ordering over the set of all probability distributions cannot be represented by a function of mean and variance, i.e., a utility function of the type $U(E, V)$.

I believe that Tsiang accepts this conclusion, and that he will take the example as an illustration that a mean-variance representation may not always give a fair approximation. He argues that in such cases we have to consider the skewness of the two distributions. Gamble *B* has a negative skewness, and hence it must be less attractive than the symmetric gamble *C*. This argument just passes the buck to authors with more patience for elaborate arithmetic. It is easy to escalate the example. For two gambles alleged to be equivalent, we can compute the mean-variance-skewness triples, and then construct a third gamble with the same triple as one of the original games, and clearly superior to the other one.

I shall not take the reader through the arithmetic involved in such escalating. Instead I shall present another classroom example, which may give some further insight into mean-variance analysis. The *gamma* density

$$g_n(x) = \frac{\alpha^{n+1}}{n!}(x-c)^n e^{-\alpha(x-c)} \qquad c \leq x$$

has the mean $E=(n+1/\alpha)+c$ and the variance $V=n+1/\alpha^2$.

Let us consider two points in the EV plane, (E_1, V_1) and (E_2, V_2) which are alleged to be on the same indifference curve. We can then construct two *gamma* distributions $g_{n_1}(x)$ and $g_{n_2}(x)$, corresponding to these points, by taking:

$$c = \frac{E_1V_2 - E_2V_1}{V_2 - V_1}$$

$$\alpha = \frac{E_2 - E_1}{V_2 - V_1}$$

$$n_1 = \left(\frac{E_2 - E_1}{V_2 - V_1}\right)^2 V_1 - 1$$

$$n_2 = \left(\frac{E_2 - E_1}{V_2 - V_1}\right)^2 V_2 - 1$$

These two distributions cannot possibly represent equivalent gambles. If $n_2 > n_1$, we have

$$(1) \qquad \int_c^z g_{n_1}(x)dx > \int_c^z g_{n_2}(x)dx \qquad \text{for all } z \text{ in } (c, \infty)$$

This implies that $g_{n_2}(x)$ is superior to $g_{n_1}(x)$. For any z, $g_{n_2}(x)$ offers the greater probability of a gain exceeding z. If n_1 and n_2 are integers, (1) follows from straightforward evaluation of the integrals. For a more general proof, consider:

$$\int_c^z \{g_{n_1}(x) - g_{n_2}(x)\}dx$$

$$= \int_c^z \left\{\frac{\alpha^{n_1+1}}{n_1!}(x-c)^{n_1} - \frac{\alpha^{n_2+1}}{n_2!}(x-c)^{n_2}\right\} \cdot e^{-\alpha(x-c)}dx$$

$$= \int_0^y \left\{\frac{x^{n_1}}{n_1!} - \frac{x^{n_2}}{n_2!}\right\} e^{-x}dx = F(y)$$

We have $F(o) = F(\infty) = 0$, and further

$$F'(y) = \frac{y^{n_1}}{n_1!}\left\{1 - \frac{n_1!}{n_2!} y^{n_2-n_1}\right\} e^{-y}$$

Hence with increasing y, $F(y)$ will increase from 0 to a maximum, and then decrease to zero as y goes to infinity. This implies that $F(y) > 0$ for any finite, positive y.

It is worth noting that the skewness of $g_n(x)$ is $2(n+1)^{-\frac{1}{2}}$. Hence with increasing n, the skewness will decrease, whilst $g_n(x)$ itself becomes more attractive. Tsiang's conclusion that skewness is an attractive property can not be generally valid.

It may be worthwhile trying to clarify a question which frequently has confused discussions about the validity of the mean-variance approach to portfolio analysis. Let us consider a set of stochastic variables with distributions belonging to a family determined by two parameters, say $F(x, \alpha, \beta)$. There will be a one-to-one correspondence between the two parameters and the mean and the variance, except in degenerate cases. We will, therefore, lose little by writing $F(x, E, V)$. A preference ordering over this set of stochastic variables will be a preference ordering over the set of ordered pairs (E, V), and it can be represented by a set of indifference curves in an EV plane. Assume now that x_1 and x_2 belong to the set under consideration, and that their distributions are $F(x, E_1, V_1)$ and $F(x, E_2, V_2)$, respectively. A portfolio of these two "assets" is then defined by a stochastic variable $z = kx_1 + (1-k)x_2$, with $0<k<1$. The mean and the variance of z are:

$$E = kE_1 + (1-k)E_2$$

$$V = k^2V_1 + (1-k)^2V_2$$

The distribution of z will not in general be $F(x, E, V)$, but a distribution of a different type, say $G(x, E, V)$. This distribution will have no place in our original preference ordering. If we give it a place by requiring that $G(x, E, V)$ shall be equivalent to $F(x, E, V)$ for any E and V, we will run into contradictions of the kind illustrated by the examples given above. G will be of the same type as F, only if F belongs to the class of distributions called "stable." The only member of this class, which has a finite variance, is the normal distribution. Hence portfolio analysis based on mean-variance methods can be applied without restriction only when all assets are represented by normally distributed variates. It may also be worthwhile recalling that a probability distribution is not uniquely determined by its moments. A textbook example (see William Feller, p. 224) is

$$(2) \quad f(x) = \frac{1}{24}\{1 - \alpha \sin \sqrt[4]{x}\} \exp(-\sqrt[4]{x})$$

which represents a probability density over $(0, \infty)$ if $0<\alpha<1$. The nth moment is $m_n=(4n+3)!/6$ which is independent of α. Hence, if we seek a preference ordering based on mean, variance, skewness and higher moments, we start with a behavioral assumption to the effect that all distributions of the form (2) are considered as equivalent. This assumption can be rejected out of hand, and so can mean-variance analysis. In spite of this, I shall continue to use mean-variance analysis in teaching, but I shall warn students that such analysis must not be taken seriously and applied in practice.

REFERENCES

K. Borch, "A Note on Uncertainty and Indifference Curves," *Rev. Econ. Stud.*, Jan. 1969, *36*, 1–4.

W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, New York 1966.

H. M. Markowitz, "Portfolio Selection," *J. Finance*, Mar. 1952, *7*, 77–91.

———, *Portfolio Selection—Efficient Diversification of Investments*, New York 1959.

J. Tobin, "Liquidity Preference as Behavior Towards Risk," *Rev. Econ. Stud.*, Feb. 1958, *25*, 65–85.

S. C. Tsiang, "The Rationale of the Mean-Standard Deviation Analysis, Skewness Preference, and the Demand for Money," *Amer. Econ. Rev.*, June 1972, *62*, 354–71.