BEYOND MEAN-VARIANCE: RISK AND PERFORMANCE MEASURES FOR PORTFOLIOS WITH NONSYMMETRIC RETURN DISTRIBUTIONS

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October 25, 1997

Revised October 26, 1998

The author thanks Steinar Ekern, Wayne Ferson, Mark Grinblatt, Ronald Kahn, Mark Rubinstein, Aamir Sheikh, and Dan Stefek for helpful discussions. Errors and omissions remain the author's fault.

Abstract

Most practitioners measure investment performance based on the CAPM, determining portfolio "alphas" or Sharpe Ratios. But the validity of this analysis rests on the validity of the CAPM, which assumes either normally distributed (and therefore symmetric) returns, or mean-variance preferences. Both assumptions are suspect: even if asset returns were normally distributed, the returns of options or dynamic strategies would not be. And investors distinguish upside from downside risks, implying skewness preference. This has led to the adoption of **ad hoc** criteria for measuring risk and performance, such as "Value at Risk" and the "Sortino Ratio."

We consider a world in which the market portfolio (but not necessarily individual securities) has identically and independently distributed (i.i.d.) returns. In this world the market portfolio will be mean-variance inefficient and the CAPM alpha will mismeasure the value added by investment managers. The problem is particularly severe for portfolios using options or dynamic strategies. Strategies purchasing (writing) fairly-priced options will be falsely accorded inferior (superior) performance using the CAPM alpha measure.

We show how a simple modification of the CAPM beta can lead to correct risk measurement for portfolios with *arbitrary* return distributions, and the resulting alphas of all fairly-priced options and/or dynamic strategies will be zero. We discuss extensions when the market portfolio is not assumed to be i.i.d.

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I. Introduction

How can one determine whether an investment manager has added value relative to risk? A correct performance assessment requires both good theory, to determine the proper measure of risk, and appropriate statistical techniques to quantify risk magnitudes. This paper focuses on measures of risk and their implications for investment performance evaluation.

While there have been some notable recent advances in the theory of performance measurement, most practice is still firmly rooted in the approach of the Capital Asset Pricing Model (CAPM).[1] In the CAPM world, the appropriate measure of risk of any asset or portfolio p is given by its "beta":

$$\textbf{(1)} \qquad b_p = \frac{Cov[r_p, r_{mkt}]}{Cov[r_{mkt}, r_{mkt}]} = \frac{Cov[r_p, r_{mkt}]}{Var[r_{mkt}]}$$

where r_p and r_{mkt} are the random returns on the portfolio p and on the market, respectively, and r_f is the riskfree rate of interest. In equilibrium, all assets and portfolios will have the same return after adjustment for risk, implying

[1] Sharpe, Alexander, and Bailey [1995] provides a good overview of current practice in Chapter 25. Grinblatt and Titman [1989] review some key issues and provide extensions of traditional alpha measurement. Glosten and Jagannathan [1994] provide an elegant and general framework. But applications of their approach required assumptions similar to our framework below (lognormal index returns and Black-Scholes option pricing), while requiring greater complexity of implementation.

(2) $$E[r_p] = r_f + \beta_p(E[r_{mkt}] - r_f)$$

Superior performance in the CAPM world is measured by "alpha", which is the incremental expected return resulting from managerial information (e.g. stock selection or market timing). This can be represented formally as

(3) $$\begin{aligned} \alpha_p &= E[r_p|M] - E[r_p] \\ &= E[r_p|M] - \beta_p(E[r_{mkt}] - r_f) - r_f \end{aligned}$$

where $E[r_p \mid M]$ is the conditional expected return to the portfolio given the information M used by manager.[2] In the CAPM equilibrium, alphas will be zero unless a manager has superior information. A portfolio with positive alpha offers an expected return in excess of its equilibrium risk-adjusted level and in this sense has superior performance.[3]

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[2] Measuring conditional expectations when managerial information is not directly observed is an important econometric challenge. Early CAPM-based studies (e.g. Jensen [1969]) regressed portfolio excess return on market excess return. The constant term was interpreted as the alpha of in our equation (3), and the slope coefficient as beta in our equation (1). Roll [1978] indicates the unreliability of alpha measures when the market portfolio proxy is not mean-variance efficient. Further difficulties in using alpha as a performance measure when managers are able to successfully time the market are discussed by Dybvig and Ross [1984]; their results are closely related to the negative state prices observed in the CAPM by Dybvig and Ingersoll [1982]. Grinblatt and Titman [1989] propose to solve the problem by using positive period-weighting measures (i.e. state price densities), although their later empirical study (Grinblatt and Titman [1994]) suggests this makes little difference for evaluating mutual fund portfolios. Ferson and Schadt [1996], while retaining the CAPM framework, argue that beta should be estimated conditionally on a vector of relevant publicly-available information variables which may change through the sample period.

[3] A related but not identical performance measure is the Sharpe ratio (SR) of a portfolio *p*, where

$$SR_p = \frac{E[r_p|M] - r_f}{\sigma_p}$$

The Sharpe ratio provides an appropriate measure of investor welfare when the investor has mean-variance preference and invests in the portfolio (and perhaps a risk-free asset) *exclusively*. Alpha, on the other hand, is a measure of performance

Underlying the CAPM and its associated risk and performance measures are strong assumptions: that either (i) all asset returns are normally (and therefore symmetrically) distributed; or (ii) investors care only about the mean and variance of returns, implying that upside and downside risks are viewed with equal distaste. Unfortunately, neither assumption justifying the CAPM approach is satisfactory. Portfolio returns are not in general normally distributed. Even if the underlying assets' returns were normal, the returns of portfolios that use options on these assets, or use dynamic strategies, will not be.[4]

Furthermore, investors typically distinguish between upside and downside risks. For example, most investors have a preference for positively skewed returns, implying that more than the mean and variance of returns is priced in equilibrium.[5]

Thus the basic underpinnings of the CAPM are suspect. Its risk measure beta is perforce equally dubious. When beta doesn't correctly measure risk, estimates of alpha will be incorrect and the performance of portfolio managers will be mismeasured. Some portfolios which offer fair (i.e. equilibrium) returns for risk will be rated as offering superior performance; others will be rated as inferior. While these shortcomings have been cited in the academic literature, the CAPM is still widely

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when the portfolio is a small part of the investor's entire (fully-diversified) portfolio of assets. A portfolio with a Sharpe ratio greater than the market's will have a positive alpha, but the converse does not necessarily hold.

[4] Rubinstein and Leland [1981] elucidate the relationship between options and equivalent dynamic strategies. Henriksson and Merton [1985] examine the relationship between options and market timing strategies.

[5] Skewness preference implies a positive third derivative of the investor's utility function, unlike the quadratic utility function which has zero third (and higher order) derivatives. An investor whose risky investments increase as wealth increases must have a positive third derivative: see Pratt [1964] and Arrow [1963]. Furthermore, Dybvig and Ingersoll [1982] observe that quadratic utility implies that (very) high-return states will have negative marginal utility and therefore negative state prices, contradicting the no-arbitrage condition of equilibrium prices. Kraus and Litzenberger [1976] extend the CAPM to the case where investors have a cubic utility function and hence skewness preference. We show below that when the market portfolio has i.i.d. returns, the average investor

used by practitioners.[6]

This paper takes a practical step beyond the mean-variance framework of performance measurement. We develop a simple risk measure that requires no more information to implement than the CAPM, but correctly captures all elements of risk, including skewness, kurtosis, and higher order moments. Thus, the results apply to *nonsymmetric* return distributions. Our model requires only two assumptions:

(i) Returns of the market portfolio are independently and identically distributed (i.i.d.) at each moment in time;

(ii) Markets are "perfect": there are no transactions costs or taxes, prices reflect perfect competition, and all relevant risks are traded in the market.

Assumption (i), while clearly strong, underlies most econometric studies and therefore is an assumption implicit in the current risk measures of practitioners. Section V considers extensions of this assumption. Assumption (ii) underlies the CAPM as well, and most other equilibrium models of asset valuation.

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must have skewness preference.

[6] This is due in part to the fact that empirical studies of alternative models (e.g. Kraus and Litzenberger [1976], Grinblatt and Titman [1994]) exhibit minimal differences from CAPM results when applied to typical stock portfolios. As our results in Section IV show, substantial differences will be evident only for portfolios or assets with highly skewed return distributions.

In the limit, as the periods become infinitesimal in length, assumption (i) implies that the market portfolio's returns will be lognormally distributed over any finite interval.[7] In continuous time the rate of return process will be a diffusion with constant drift and volatility, and therefore consistent with the models of Black and Scholes [1973] and Merton [1973].[8]

Observe that we are *not* assuming that individual asset or portfolio returns are lognormal: our assumption of i.i.d. returns and the resulting lognormal return distribution refers *only to the market portfolio*.[9] Note also that we are not (directly) assuming any particular utility function as representing investor preferences.

We seek a valid risk measure for portfolios--both with and without derivatives--which have *arbitrary* distributions of returns. The correct risk measure should have the property that any portfolio strategy has zero measured excess returns after adjustment for risk, if that strategy can be implemented without superior information.

Section II shows that, given assumptions (i) and (ii) alone, the market portfolio will *not* be mean-variance efficient over any finite time interval: a dynamic strategy which does not require superior managerial information will have a higher Sharpe ratio than the market, and therefore a positive CAPM alpha. Furthermore, equation (2) no longer holds: the CAPM beta does not properly

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[7] The usual central limit theorem conditions are required. In a recent empirical examination of market returns 1928-1996, Jackwerth [1997] finds that while daily market returns are not lognormal, over longer periods (e.g. 3 months) returns are quite "close" to lognormally distributed.

[8] Lognormality results from a continuous diffusion process for the rate of return if both the drift and volatility of the process are constant. While requiring constant volatility, Black and Scholes' model does not require that the drift of the asset rates of return be constant, and therefore distributions other than the lognormal may be consistent with their model.

[9] It is well known that a portfolio of assets with lognormal returns will not itself have lognormal returns. However, we are not assuming that lognormality holds for every asset, but rather for the market alone.

measure risk. This in turn implies that the CAPM alpha incorrectly measures performance. Mismeasurement is particularly pronounced for portfolios with highly skewed returns, such as those using options or following dynamic strategies.

We show that strategies that sell fairly-priced options on the market, or increase market exposure after market declines, will be accorded positive CAPM alphas; strategies that buy options or decrease market exposure after market declines will have negative CAPM alphas. With proper risk measures these strategies should be accorded a zero alpha, since they do not require additional managerial information about asset returns in order to be implemented.

The CAPM's failure to assess performance correctly results from the fact that skewness matters under assumptions (i) and (ii). Even though the assumptions do not directly presume skewness preference, we show that they *imply* that the market places a positive value on skewness. And skewness preference in turn implies that upside risks are less important to investors than downside risk.[10]

If the CAPM is incorrect when the market portfolio has i.i.d. returns, does there exist a correct measure of risk? In Section III, we show that the answer to this question is affirmative. A relatively straightforward modification of the CAPM beta provides a valid risk measure for any asset, portfolio, or dynamic strategy. This modified beta requires no more data to estimate than does the CAPM beta.

Section IV shows that the differences between the correct beta and the CAPM beta are small,

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[10] As an *ad hoc* approach to recognizing the greater importance of downside risk, Sortino and Vandermeter [1991] have proposed that the Sharpe ratio be modified by replacing the variance of returns in the denominator with the lower semi-variance of returns. A related risk measure is "Value at Risk", the loss which could occur over a fixed time period with small probability, e.g. 1%. These approaches are not grounded upon capital market equilibrium theory, and may themselves spuriously identify superior or inferior managerial performance. See also Kahn and Stefek [1997].

and the mismeasurement of alphas will be similarly small, for assets or portfolios whose returns are jointly lognormal with the market. The correct beta differs substantially from the CAPM beta for portfolio or asset returns which are highly skewed, and thus becomes critical for the performance measurement of investment strategies using options, market timing strategies, or other dynamics.

Finally, Section V briefly discusses the correct risk measure when assumption (i) does not hold, and the market return follows a stochastic process which is not i.i.d.

II. Problems with Mean-Variance Performance Measures in an i.i.d. World

Below we develop a simple 2-period i.i.d. binomial example which shows that *the market portfolio is mean-variance inefficient.* We demonstrate that there exists a simple dynamic strategy that does not require superior information to implement, but has a higher Sharpe ratio than the market portfolio.

II(i). A Simple Binomial Example

Let the market portfolio increase by 25% or fall by 20% each year over a 2 year period. The probability of an up move is assumed to be 80%, giving the market an annual expected return of 16%. The standard deviation of the market returns over the two-year period is 29.71%. The annual riskfree rate is assumed to be 5%, implying a Sharpe ratio over the two year period of $[1.16^2 - 1.05^2]/[.2971] = 0.8182$. It is easily shown that the static strategy which puts half its initial wealth in the risky portfolio, and half in bonds has the same Sharpe ratio, with an expected return of 22.40% and standard deviation of 14.85% over the two year period.

Now consider the following dynamic strategy: start with a 60/40 stock-to-cash investment ratio. If the market rises in the first period, sell 44.8% of the stock holding and convert it into cash. (Beginning the second period, 35.4% of holdings will be in stock in stock, and the remaining fraction in cash). If the market falls in the first period, liquidate all cash holdings and invest them in stock (beginning the second period, 100% of holdings will be in stock). After two years, this dynamic strategy will have an expected return of 22.80% and a standard deviation of 13.48% over the two-year period. The former is higher than, and the latter is lower than, the 50/50 static strategy. The Sharpe ratio is .9310, substantially higher than that of the market or the 50/50 strategy. And a higher Sharpe ratio than the market implies a positive CAPM-measured alpha.

While the above strategy is multi-period (and therefore inconsistent with a single-period CAPM), there exists a *static* strategy using fairly-priced options that yields *exactly the same result as the dynamic strategy* above. For each dollar of initial wealth, the option-based strategy would sell 0.8 fairly priced at-the-money 2-period call options on the risky asset and invest all initial wealth, plus the receipts from selling the call options, in the risky asset. It would not subsequently change its portfolio holdings. The interested reader can verify that this strategy yields the same payoffs as the dynamic strategy in each future state of the market.

By leveraging the dynamic strategy, or its options equivalent, a higher expected return and lower risk than the market portfolio can be obtained. The simple assumption of i.i.d. market returns therefore implies that the market portfolio is mean-variance inefficient in a perfect capital market![11]

II(ii). Analysis of the Example

The mechanistic dynamic strategy above appears to "beat the market." Under traditional CAPM-based measures, it would be accorded superior performance, although anyone could follow such a strategy.

The intuition underlying our example is the following. It has been shown elsewhere (Rubinstein [1976], Brennan [1978], He and Leland [1993]) that if the market portfolio's rate of return is i.i.d. and markets are perfect, the representative investor (whose preferences determine all prices) *must* have a power utility function.[12] This utility function has a positive third derivative, implying skewness preference: skewness will be positively valued by the market. Any investor can improve her performance in mean-variance terms by "selling" skewness, i.e. by accepting negatively skewed returns in return for improvements in mean and/or variance. This is exactly what the dynamic strategy in our example creates: negative skewness relative to the market return. If only mean and variance are assessed, the negatively-skewed returns will seem to "outperform".[13] Outperformance is a misnomer

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[11] In the binomial model, it can be shown that the market is M-V efficient over each subperiod. (Hint: in a two-state world, any option on the market portfolio can be perfectly replicated by a static portfolio of the market and the riskfree asset). But since we assume that the sub-periods can be arbitrarily short (in the limit becoming a logarithmic random walk), the market will always be M-V inefficient over any finite interval.

[12] In the continuous time limit, markets are dynamically complete (Harrison and Kreps [1979]) and a representative investor exists even when individual investors have heterogeneous utility functions (Constantinides [1982]).

[13] The example does not give the highest possible Sharpe ratio. In continuous time, assume the market rate of return process has drift μ and volatility s, and consider a mean-variance investor (who has quadratic utility) with

here, in the sense that the average investor would *not* prefer to sacrifice skewness to improve in terms of mean and variance only. Nor, as discussed above, does the CAPM-based "outperformance" mean that the investment manager has added value by identifying undervalued assets or by informed market timing.

II(iii) The performance of strategies using options on the market

A closely-related implication of the above discussion is that portfolios which contain fairly-priced option positions (or follow equivalent dynamic strategies) also will have their performance mismeasured. We consider two classes of option strategies: those which write a call option on the market against an underlying position in the market portfolio, and those which buy a put option. Option strike prices range from deep "in-the-money" to deep "out-of-the-money." We assume the market follows a logarithmic Brownian motion with annual expected return of 12%, and annual volatility of 15%.[14] The riskfree rate is 5%. Since this is a Black-Scholes world, the option prices will be determined by the Black-Scholes formula.[15] It is straightforward to compute the expected returns, covariances with the market, and CAPM beta of any option-based strategy using these parameters and the lognormality of the market return.

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satiation wealth level equal to *k*. Then it can be shown that at any time *t* the investor's optimal strategy is to invest a fraction $a(t)$ of wealth $W(t)$ in the market portfolio, where $a(t) = [(\mu - r)/s^2][k/W(t) - 1]$, for $W(t) \leq k$. Bajeux-Besnainou and Portait [1993, revised 1995] further show that when there are many risky securities, all dynamic mean-variance efficient strategies are buy-and-hold combinations of two funds: a continuously rebalanced portfolio of these securities, and a zero-coupon bond with maturity equal to the investor's horizon.

[14] The lognormal distribution parameters are μ_m = 10.44%, s_m = 13.33%.

[15] Rubinstein [1976] shows that the Black-Scholes formula correctly prices options on the market in discrete time, when market returns are lognormally distributed and the representative investor has power utility.

The first class of option strategies, holding the market portfolio and writing one-year covered calls on the market, creates payoffs which are a concave function of the market payoff, and thereby reduces or "sells" skewness. The dynamic strategies equivalent to writing covered calls have the feature that they sell the market portfolio as its price rises, and buy as its price falls, without superior information. We (loosely) label this class "rebalancing" or "value" strategies. Columns (i), (ii), and (iii) of Panel A in Table I lists the annual expected return, CAPM beta, and CAPM alpha of strategies which write one-year calls at different strike prices.

The second class of option strategies, which buys put options on the market, creates convex payoffs and therefore creates or "buys" additional skewness. We (again loosely) label these as "momentum" or "portfolio insurance" strategies. The equivalent dynamic strategy buys the market portfolio on strength and sells on weakness. Columns (i), (ii), and (iii) of Panel B in Table I lists the expected return, CAPM beta, and CAPM alpha of strategies which buy one-year put options at different strike prices.

When skewness is positively valued, mean-variance based performance measures will overrate the rebalancing strategies which reduce skewness, and underrate the momentum or portfolio insurance strategies which buy skewness.[16] Figure I, based on Columns (i) and (ii) of Table I, plots the expected returns and CAPM betas of the two classes of option strategies, for different strike prices. The rebalancing or value strategies, which plot above the security market line (joining the riskless asset and

[16] While Dybvig and Ingersoll [1982] suggested that call options could be underpriced due to the negative marginal utility of the quadratic utility function at high levels of wealth, our argument suggests that call options could be underpriced by the CAPM even if portfolio returns were bounded to levels of wealth less than the satiation level. Call options have greater skewness than the market, and would be undervalued by CAPM measures which ignore the positive value of skewness.

the market portfolio), hold the market portfolio and sell a fairly-priced one year call option on the market. Momentum or portfolio insurance strategies, which plot below the security market line, hold the market portfolio and buy a fairly-priced 1 year put option on the market.[17] In both cases, option strike prices range from 90 to 140 percent of the current market value.

CAPM-based alphas are measured by the vertical distance between the point representing each portfolio and the security market line, and are listed in column (iii) of Table I. Alphas are substantially different from zero for strike prices near the money.[18]

Of course, properly measured alphas here *should* be zero: options are assumed to be purchased at a fair market price. They are not zero because the CAPM risk measure beta is incorrect, and equation (2) does not hold when the market is lognormally distributed. Although the manager has no additional information (i.e. $E[r_p|M] = E[r_p]$), α_p in equation (3) is nonzero. Note that any investment manager can "game" the CAPM performance measurement by selling options or rebalancing.

While our examples consider strategies buying or selling options on the market, similar results are likely when individual security options are bought or sold, since these strategies will also affect the skewness of the managed portfolio relative to the market.

III. Correct Measures of Risk and Performance

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[17] Bookstaber and Clarke [1985], while not providing analytical results, observed from simulations that option-based strategies seemed to lie above or below the CAPM "market line".

[18] When naked options on the market portfolio are considered, the mismeasurement becomes even more extreme. For example, a 1 year call option on the market with strike price 110% of the current market value (and parameters as in Table I) has a CAPM beta of 17.88, whereas its modified beta is 14.32. A CAPM-based analysis of a naked option position (or dynamics replicating this position) would indicate a negative annual alpha of 25%!

We have shown that the CAPM-based alpha systematically mismeasures performance when the market has i.i.d. returns.[19] This is because the CAPM-based beta, the measure of an asset's risk, does not capture skewness and other higher-order moments of the return distribution which investors value. The first "patch" might be to incorporate skewness, as in Kraus and Litzenberger [1976]. But this is insufficient, since the power utility function consistent with a lognormally-distributed market has non-zero derivatives of all order. That is, kurtosis also matters to investors, as do even higher order moments of the return distribution.[20] Any risk measure in this world must capture an *infinite* number of moments of the return distribution--a daunting task!

Fortunately, past research has examined a closely related problem. Rubinstein [1976] considers asset pricing in a model with power utility functions and lognormal returns for the market portfolio, both of which are implied by our assumptions (i) and (ii). He derives an equilibrium pricing equation which holds for assets with any returns over some time interval:[21]

(4) $$P_0 = \frac{E[(1+r_p)P_0] - l r[(1+r_p)P_0, -(1+r_{mkt})^{-b}]\, Std[(1+r_p)P_0]}{1+r_f}$$

[19] See He and Leland [1993] for a discussion of the (unreasonable) stochastic process of the market which would be required for the CAPM to evaluate risk correctly.

[20] Indeed, it is readily observed that the derivatives of the power utility function alternate in sign. Thus, mean, skewness, and higher odd-numbered moments of the distribution are always positively valued by investors; variance, kurtosis, and higher even-numbered moments are negatively valued.

[21] Rubinstein [1976], equation (3). Actually there is a misprint in Rubinstein's equation: the numerator contains a covariance which correctly should be a correlation. Rubinstein's equation (2), from which (3) is derived, has the correct term. The Rubinstein [1976] result is closely related to the general single-period result he derives in Rubinstein [1973].

where P_0 is the price of any asset, r_p and r_{mkt} are the returns to the portfolio and market over the time interval, $?[x, y]$ is the correlation of x and y, $-b < 0$ is the exponent of the marginal utility function of the average investor, and

(5)
$$l = \frac{Std[(1+r_{mkt})^{-b}]}{E[(1+r_{mkt})^{-b}]}$$

Dividing both sides of equation (4) by P_0, rearranging terms, and using the fact that this equation must also hold for the market portfolio gives

(6)
$$E[r_p] = r_f + B_p(E[r_{mkt}] - r_f)$$

where

(7)
$$B_p = \frac{Cov[r_p, -(1+r_{mkt})^{-b}]}{Cov[r_{mkt}, -(1+r_{mkt})^{-b}]}$$

Furthermore, Rubinstein [1976] and Breeden and Litzenberger [1978] show how the exponent b is related to the excess return of the market, when the market is lognormally distributed:

(8)
$$b = \frac{\ln(E[1+r_{mkt}]) - \ln(1+r_f)}{Var[\ln(1+r_{mkt})]}$$

This coefficient is a "market price of risk": the market's instantaneous excess rate of return divided by the variance of the market's instantaneous rate of return.[22]

Parallel to the CAPM-based alpha, the appropriate measure of excess returns A_p will therefore

[22] In continuous time, $b = (\mu_{mkt} - r_f)/s_{mkt}^2$, where the market portfolio process is $dM/M = \mu_{mkt}dt + s_{mkt}dz$.

be

$$A_p = E[r_p|M] - B_p(E[r_{mkt}] - r_f) - r_f \quad \textbf{(9)}$$

Notice that A_p differs from α_p in equation (3) only because our measure of risk B_p differs from β_p. But clearly β_p and B_p are related, as a comparison between equations (1) and (7) shows. Furthermore, the estimates of A_p and B_p require no more raw data than the estimates of α_p and β_p of the CAPM-based model.[23] The coefficient B_p depends on the covariance of the portfolio return and one plus the market return raised to the $-b$ power. The coefficient b depends upon the market return mean and variance and the riskfree rate, parameters which are required by the CAPM as well.

Table 1, column (iv) presents the correct risk measures B, which can be compared with the CAPM-based risk measures β in column (ii). If we replace the measure of risk β with the measure of risk B, the alphas of optioned portfolios become zero as is seen in column (v). That is, using the correct measure of risk gives the correct result, that managers who buy or sell fairly priced assets add no value!

There does not seem to be a useful general substitute for the Sharpe ratio when applied to dynamic strategies or options. But previous work by Leland [1980] and Brennan and Solanki [1981] offer some insights. Leland shows that an investor whose risk tolerance grows with wealth more quickly than the average investor will want portfolio insurance (convexity); if risk tolerance grows less quickly than the market's, a rebalancing strategy (concavity) is optimal. Risk tolerance grows more quickly when the investor has greater skewness preference. Optimal strategies therefore are preference

[23] Note that the many of the econometric problems related to estimating α_p mentioned in footnote 3 will also be relevant to estimating A_p, including finding an appropriate proxy for the market portfolio.

dependent and no measure which depends only on the distribution of portfolio returns will correctly rank all alternatives. Brennan and Solanki derive an interesting partial result, however. If rankings are limited to the set of portfolios *p* which have lognormal returns, then the best of that set should maximize $(\mu_p - r_f)/s_p$. Furthermore, amongst lognormal portfolios which could serve as the underlying portfolio for constructing nonlinear payoffs (through option or dynamic strategies), the best choice is the one which maximizes this ratio. The actual best nonlinear strategy will of course be preference-dependent.

As indicated, applying the Sharpe ratio to a portfolio with *nonlognormal* returns will in general produce nonsense as a measure of managerial ability. But this does not detract from the modified alpha (i.e. A_p) measure of performance, since that can identify a manager's ability to select underpriced assets (or correctly market time).

IV. B vs. ß: Assets with Lognormal Returns

We have shown that *B*, not ß, is the appropriate measure of risk of *any* asset or portfolio, when the market itself has lognormal returns. And we have shown that the difference between the two may be substantial, when applied to assets or portfolios whose returns are distinctly skewed, such as options or dynamic strategies.

But many portfolios and assets, including most equities, have returns which are approximately lognormal (although the return distribution's parameters may be quite different from the market portfolio's). If we use ß rather than *B* as the risk measure for such assets, are we making a major mistake? The answer is "no", if the intervals over which we make observations are one year or less. The Appendix shows that the two risk measures are closely related in the case of where portfolio and market returns are jointly lognormal.

Table II utilizes the theory developed in the Appendix to examine the difference between B_p and β_p for portfolios which are jointly lognormally distributed with the market. We observe that the deviations between β and B are relatively small, and consequently the differences between a and A are small. B tends to be slightly closer to 1 than β. Furthermore, the differences become even smaller when the time interval of observations is less than one year.[24]

Therefore it appears to matter little whether one estimates B or β to assess the performance of assets or portfolios whose returns are (approximately) jointly lognormal with the market return. Other estimation errors are likely to far outweigh the errors which result from using β rather than B. Only when portfolios have distinctly skewed returns will there be an important difference between the CAPM and modified technique in measuring performance.

V. When the Market Return is Not i.i.d.

The work of He and Leland [1993] suggests a means to extend the analysis when the market portfolio follows a diffusion process with drift and volatility components which may change with time and with the market level. (Examples would include constant elasticity of variance (CEV) or Ornstein-

[24] Subsequent empirical studies of equity portfolio betas undertaken by Aamir Sheikh of BARRA have confirmed that β and B coefficients with 3-month and 6-month measurement periods are practically identical. Grinblatt and Titman [1994] also find that performance evaluations of mutual fund returns are relatively insensitive to using the CAPM or power (marginal) utility approach.

Uhlenbeck mean-reverting processes). He and Leland show how to derive the representative investor's utility function which supports a given market stochastic process.

Knowledge of the representative utility function then allows Rubinstein's [1973] result that the appropriate risk adjustment (or modified "beta") for a portfolio is the ratio of the covariance of the portfolio's return with marginal utility divided by the expected covariance of the market portfolio's return with marginal utility.

It would be surprising if the market utility function derived from the market's stochastic process did not exhibit skewness preference (see footnote 5 above). If this is the case, it continues to follow that the CAPM approach will over- (under-) value strategies which exhibit negative (positive) co-skewness with the market return. Thus the qualitative nature of our earlier results will hold in a much more general environment: call-write or rebalancing strategies will typically be overrated given by CAPM performance measures, whereas portfolio insurance or momentum strategies will be underrated. As before, the more pronounced the change in skewness relative to the market return, the worse the CAPM performance measures will be.

VI. Conclusion

The simplest possible assumption about market rates of returns is that they are identically and independently distributed (i.i.d.). Under weak assumptions, the market return will be lognormally distributed as the number of compounded i.i.d. subperiods becomes large.

Remarkably powerful results follow from market lognormality. Under the perfect market assumption (ii), the average investor will have a power marginal utility function, which in turn can be used to derive equilibrium asset prices. This in turn provides a measure of risk (our *B*) which determines the required return of any fairly priced asset or portfolio strategy, including those with highly nonsymmetric return distributions. Superior or inferior performance (our *A*) is the expected return based on managerial information, less the required return.

Our risk measure differs substantially from the CAPM beta, when asset or portfolio returns are highly nonlinear in the market return. Correctly measuring risk is essential for assessing the performance of an investment manager when options are used, or when dynamics (including market timing strategies) create nonlinear payoffs. The difference in between *B* and β, however, will be relatively small when the portfolio or asset returns are jointly lognormal with the market.

Other measures, such as the "Sortino ratio" or "Value at Risk", are *ad hoc* attempts to incorporate the importance of downside risk. But as they totally ignore upside risk, they are generally inaccurate as a appropriate risk and/or performance measures. Our measure is exact for any distribution of asset or portfolio returns, as long as the market return is i.i.d.

What if the market return is not lognormally distributed? *If* we can estimate the market's price process, we can in principle combine the results of He and Leland [1993] and Rubinstein [1973] to develop appropriate measures of risk and performance. He and Leland’s results permit identification of a marginal utility function consistent with an average investor who will "support" a given market price process. The appropriately modified beta is the covariance between the marginal utility of the average investor and the asset or portfolio return, divided by the expected covariance between the marginal utility of the average investor and the market return.

APPENDIX:

COMPARISONS OF B_p and $ß_p$

FOR LOGNORMALLY DISTRIBUTED ASSETS

Recall that B_p is defined as

$$B_p = \frac{Cov(r_p, R_M^{-b})}{Cov(r_{mkt}, R_M^{-b})} = \frac{Cov(R_p, R_M^{-b})}{Cov(R_M, R_M^{-b})}$$
$$= \frac{E(R_p, R_M^{-b}) - E(R_p)E(R_M^{-b})}{E(R_M, R_M^{-b}) - E(R_M)E(R_M^{-b})}$$

where $R_M = (1 + r_{mkt})$ and $R_p = (1 + r_p)$. If R_M and R_p are jointly lognormal, with

$E[log(R_M)] = \mu_M$, $\quad$ $E[(log(R_M) - \mu_M)^2] = s_M^2$

$E[log(R_p)] = \mu_p$, $\quad$ $E[(log(R_p) - \mu_p)^2] = s_p^2$

$Cov[log(R_M), log(R_p)] = s_{pM}$

then

$$B_p = \frac{\text{Exp}[-b m_M + m_p + .5(b^2 s_M^2 - 2b s_{pM} + s_p^2)] - \text{Exp}(m_p + .5 s_p^2)\text{Exp}[-b m_M + .5 b^2 s_M^2]}{\text{Exp}[-b m_M + m_M + .5(b^2 s_M^2 - 2b s_M^2 + s_M^2)] - \text{Exp}(m_M + .5 s_M^2)\text{Exp}[-b m_M + .5 b^2 s_M^2]}$$

Factoring numerator and denominator gives

$$B_j = \frac{\text{Exp}[-b m_M + .5 b^2 s^2_M + m_p + .5 s^2_p](\text{Exp}[-b s_{pM}] - 1)}{\text{Exp}[(1-b) m_M + .5(1+b^2) s^2_M](\text{Exp}[-b s^2_M] - 1)}$$

$$= \text{Exp}[m_p - m_M + .5 s^2_p - .5 s^2_M]\left(\frac{\text{Exp}[-b s_{pM}] - 1}{\text{Exp}[-b s^2_M] - 1}\right)$$

Now $\beta_p = Cov(r_p, r_M)/Var(r_M)$ is simply the above expression when $b = -1$.

Therefore, after simplification

$$\frac{B_p}{b_p} = \left(\frac{e^{-b s_{pM}} - 1}{e^{-b s^2_M} - 1}\right)\left(\frac{e^{s^2_M} - 1}{e^{s_{pM}} - 1}\right)$$

To a first order Taylor Series expansion, $e^x = 1 + x$. It immediately follows that, to the first order, $B_p / \beta_p = (-b s_{pM} / -b s_M^2)(s_M^2 / s_{pM}) = 1$. Over relatively short time periods (when volatilities are small), both techniques will yield identical estimates for "beta". For longer time periods, the two techniques will not give identical results: see Table II of the text.

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FIGURE 1: CAPM Plot of Option Strategies

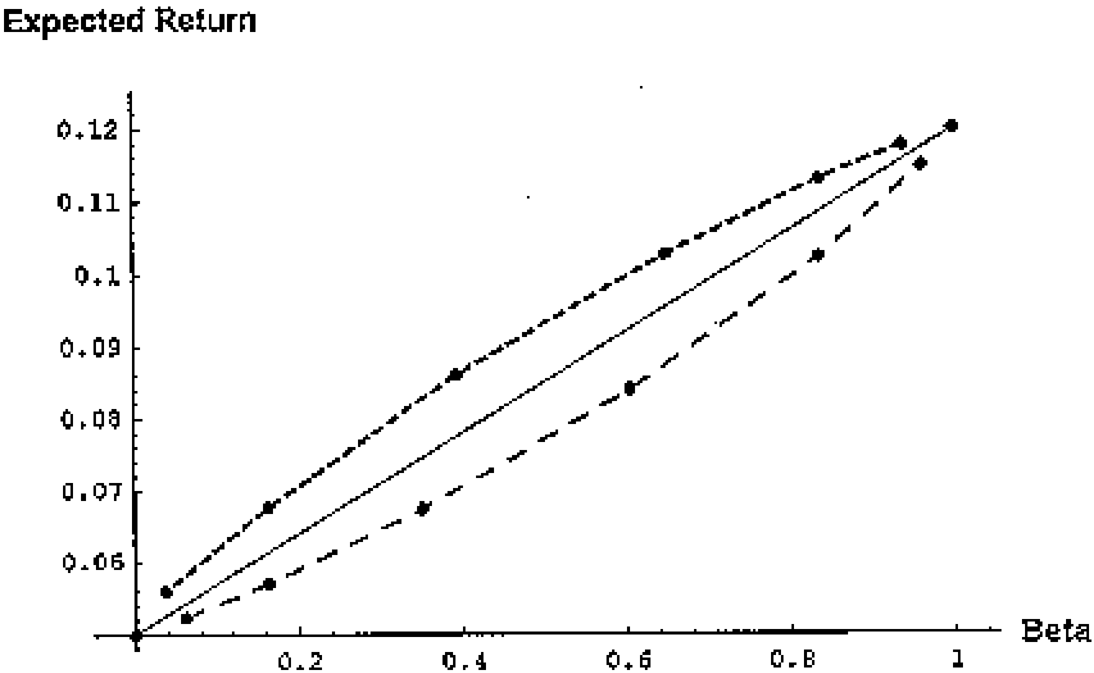

Figure 1 plots the security market line, the straight line joining the riskfree asset point (ß = 0, E(r) = 0.05) with the market portfolio point (ß = 1, E(r) = 0.12). The annual standard deviation of the market portfolio return is 15%.

The dashing line above the security market line is the plot of rebalancing or value strategies for alternative strike prices of the call option sold. The points along this line range from strike price 90 (lower left) to strike price 140 (upper right).

The large dashing line below the security market line is the plot of momentum or portfolio insurance strategies for alternative strike prices of the put option bought. The points along this line range from strike price 140 (lower left) to strike price 90 (upper right).

Alpha is measured by the vertical distance between the plotted point and the security market line.

TABLE I:

CAPM-based ß and a vs. Modified B, A

***Rebalancing or Value Strategies*:** **Long the Market; Short 1 Call**

Strike Price	*(i) E(r)*	*(ii) ß*	*(iii) a*	*(iv) B*	*(v) A*
90	**5.51%**	**.038**	**0.24%**	**.073**	**0**
100	**6.76%**	**.163**	**0.62%**	**.251**	**0**
110	**8.61%**	**.394**	**0.85%**	**.515**	**0**
120	**10.27%**	**.650**	**0.72%**	**.753**	**0**
130	**11.30%**	**.838**	**0.57%**	**.900**	**0**
140	**11.77%**	**.939**	**0.20%**	**.967**	**0**

***Portfolio Insurance or Momentum Strategies*:** **Long the Market; Long 1 Put**

Strike Price	*(i) E(r)*	*(ii) ß*	*(iii) a*	*(iv) B*	*(v) A*
90	**11.49%**	**.962**	**-0.24%.927**	**0**	
100	**10.24%**	**.837**	**-0.62%.749**	**0**	
110	**8.40%**	**.606**	**-0.84%.485**	**0**	
120	**6.73%**	**.351**	**-0.72%.247**	**0**	
130	**5.70%**	**.163**	**-0.44%.101**	**0**	
140	**5.24%**	**.062**	**-0.19%.034**	**0**	

Column (i) is computed assuming a lognormal market portfolio with annual mean = 12%, and std. dev. = 15%, and the distributions this implies for portfolios with options.

Column (ii) computes equation (1), using the assumptions in column (i).

Column (iii) computes equation (3), with $E[r_p \mid M]$ = column (i). $r_f = 5\%$.

Column (iv) computes equation (7). Equation (8) implies $b = 3.63$.

Column (v) computes equation (9).

TABLE II

Values of B_p ($ß_p$) for Lognormally Distributed Assets

		$?_{p,mkt}$		
		.25	.50	.75
	.15	.256 (.248)	.508 (.498)	.756 (.748)
s_p	.25	.415 (.405)	.819 (.813)	1.213 (1.224)
	.35	.561 (.551)	1.103 (1.108)	1.625 (1.670)

Table II assumes the portfolio p and the market portfolio returns are jointly lognormal.

The market has annual mean = 12% and std. dev. = 15%. The annual riskfree rate is 5%.

Portfolios p with differ with respect to their correlations with the market (columns), and different volatilities (rows).

Table entries are the computed B_p and, in parentheses, CAPM $ß_p$.